Updates on Traveling Salesman in Banach Spaces

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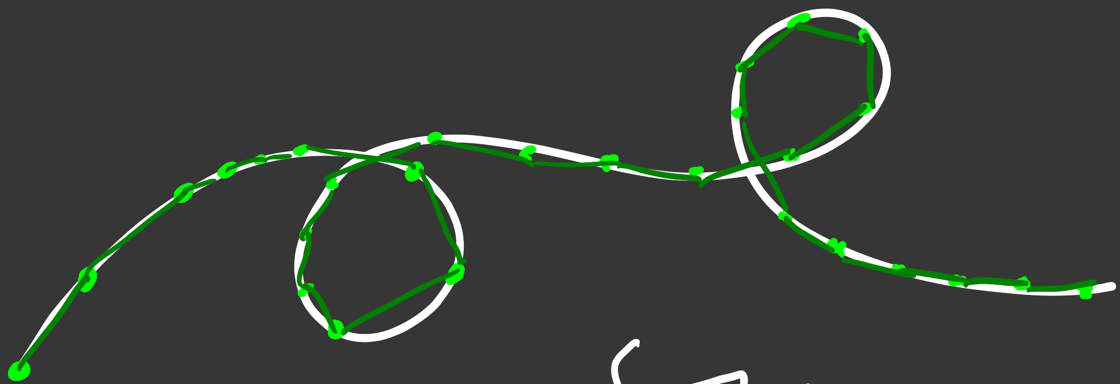
$\mathbb{X}$ metric space

$\Gamma \subset \mathbb{X}$ is a **curve** if $\Gamma = f([0,1])$

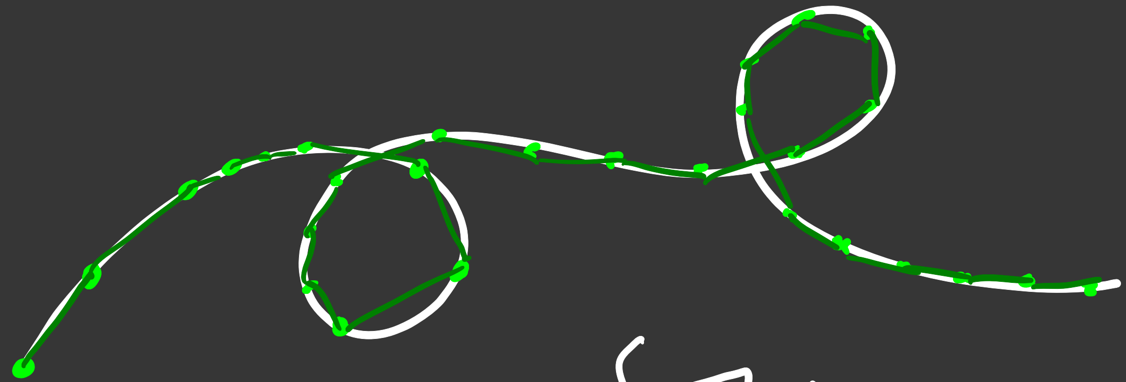
for some cts map $f : [0,1] \to \mathbb{X}$

f is a **parameterization** of Γ



"Intrisic Length"

$$\operatorname{var}(f) = \sup\left\{ \sum_i |f(x_i) - f(x_{i-1})| : \text{Partitions of } [0,1] \right\}$$



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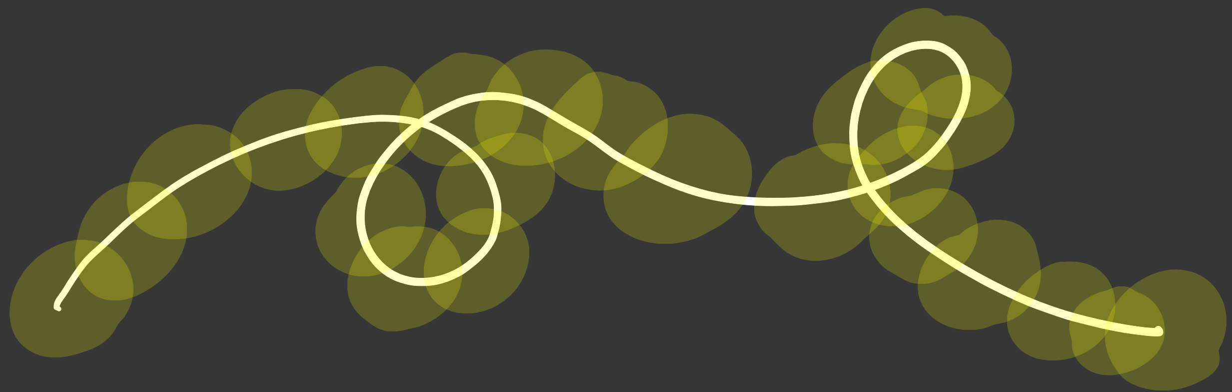


"Extrinsic Length"

$$\mathcal{H}^1(\Gamma) = \lim_{\delta \downarrow 0} \inf\left\{ \sum_i \operatorname{diam} U_i : \begin{array}{l} \Gamma \subset \bigcup_i U_i \\ \operatorname{diam} U_i \leq \delta \end{array} \right\}$$

1-dimensional Hausdorff Measure

Ważewski's Theorem

RECT

X is a metric space
$\Gamma \subset X$ nonempty

T.F.A.E.

1. Γ is a rectifiable curve, i.e. $\Gamma = f([0,1])$ for some f with $\operatorname{var}(f) < \infty$
2. Γ is compact, connected, and $\mathcal{H}^1(\Gamma) < \infty$
3. Γ is a Lipschitz curve, $\Gamma = f([0,1])$ for some f s.t. $|f(x) - f(y)| \leq L\,|x-y|$

$$\ell_p = \left\{ (x_1, x_2, x_3, \dots) \in \mathbb{R}^\omega : \sum_i |x_i|^p < \infty \right\}$$

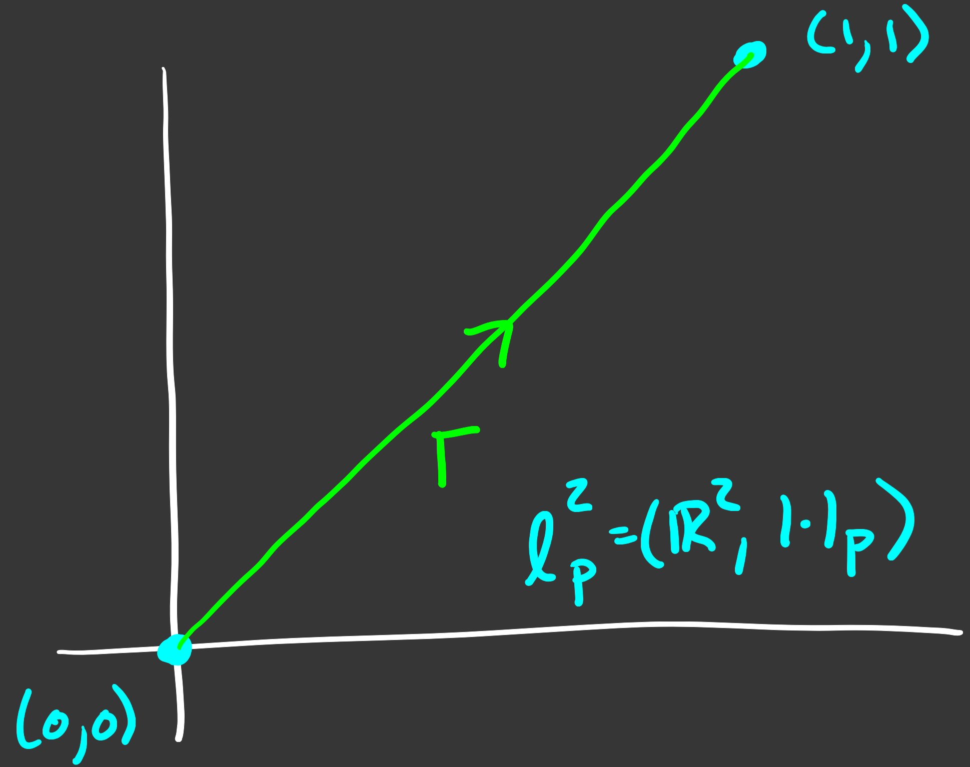
- Banach space when $1 \le p < \infty$

  $|x|_p = \left(\sum_i |x_i|^p\right)^{1/p}$, $\quad \mathrm{dist}_p(x,y) = |x-y|_p$
- separable $(1 \le p < \infty)$, reflexive $(1 < p < \infty)$
- increasing:

  $$p < q \implies \ell_p \subset \ell_q$$

  Idenity is 1-Lipschitz embedding : $|x|_q \le |x|_p$

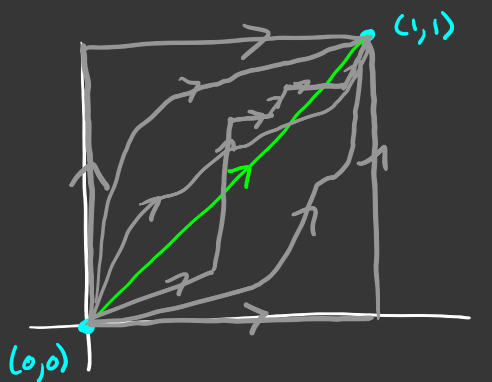
Corollary Γ rectifiable in $\ell_p \implies \Gamma$ rectifiable in ℓ_q



$$\mathcal{H}^1(\Gamma) = |(1,1)-(0,0)|_p = 2^{1/p}$$

- rectifiable in each ℓ_p
- shorter as $p \to \infty$

- In finite-dimensions, rectifiability independent of norm but length of curve depends on norm
- What about in infinite-dimensions?

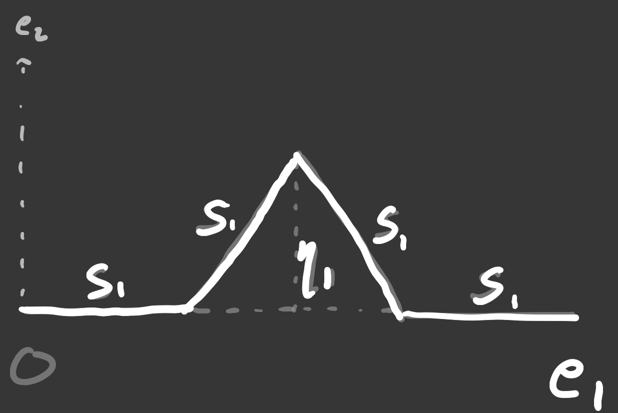


In ℓ_1, there are infinitely many geodesics between (0,0) and (1,1)

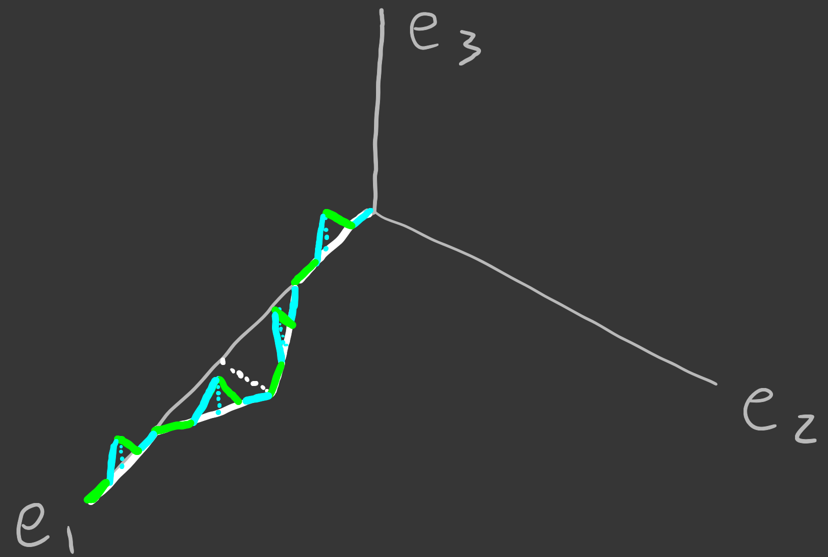
Example (B-McCurdy)

Related Example by Edelen-Naber-Valtorta

For all $1<p<\infty$, there exists a curve Γ in ℓ_p s.t.
$\mathcal{H}^1_{\ell_p}(\Gamma)=\infty$ and $\mathcal{H}^1_{\ell_q}(\Gamma)<\infty$ for all $q<p$.



Add blip of relative height η_1 in e_2-direction

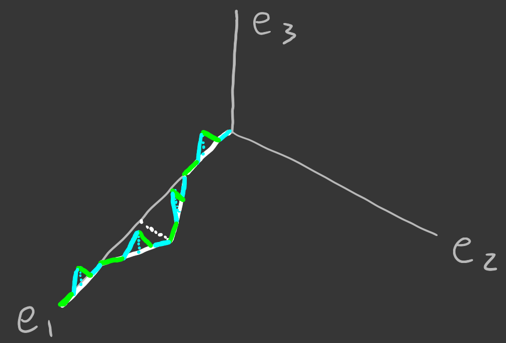


Add blips of relative height η_2 in e_3 direction

Add blips of relative height η_3 in e_4 directions . . .

Example (B-McCurdy) Related Example by Edelen-Naber-Valtorta

For all $1<p<\infty$, there exists a curve Γ in ℓ_p s.t.
$\mathcal{H}^1_{\ell_p}(\Gamma)=\infty$ and $\mathcal{H}^1_{\ell_q}(\Gamma)<\infty$ for all $q>p$



Basic Computation

$$\boxed{\begin{array}{l}\Gamma\subset\ell_p \quad \text{relative heights } \eta_i \\ \mathcal{H}^1_{\ell_p}(\Gamma)\approx\exp\left(\sum_i \eta_i^p\right)\end{array}}$$

- Rectifiable $\iff$ $\sum_i \eta_i^p<\infty$
- If rectifiable, then Γ is Ahlfors regular

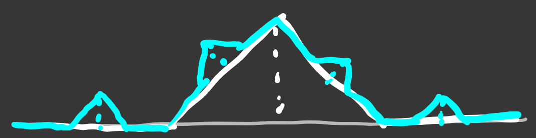
Choose $\eta_i=\dfrac{\delta}{i\log(i+i_0)}$ with $\delta>0$, $i_0\geq 1$ so $\eta_1\leq\frac{1}{16}$

$$\mathcal{H}^1_{\ell_p}(\Gamma)=\infty,\quad\text{but}\quad \mathcal{H}^1_{\ell_q}(\Gamma)\lesssim\exp\left(\sum_i\frac{\delta^{q/p}}{i\log(i+i_0)^{q/p}}\right)<\infty$$

when $p<q$

We still do not have a complete picture!

$\ell_p^2 = (\mathbb{R}^2, |\cdot|_p)$

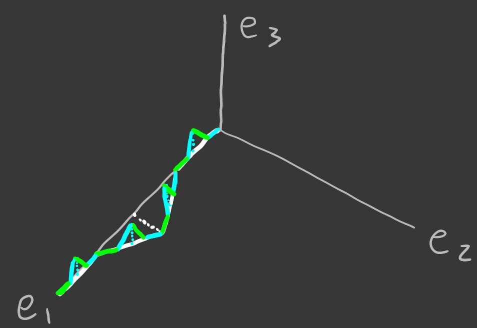


von Koch curve

- add blips in "$\perp$" directions
- relative heights η_i

$$\mathcal{H}^1_{\ell_p}(\Gamma) \approx \mathcal{H}^1_{\ell_2}(\Gamma) \approx \exp\left(\sum_i \eta_i^2\right)$$

ℓ_p infinite-dimensions



von Koch curve

- add blips in new e_{i+1} directions
- relative heights "η_i"

$$\mathcal{H}^1_{\ell_p}(\Gamma) \approx \exp\left(\sum_i \eta_i^p\right)$$

Length gained by adding blips sensative to direction of blip!

<u>Question</u>: Can you build Γ, $\mathcal{H}^1_{\ell_p}(\Gamma) \approx \exp\left(\sum_i \eta_i^q\right)$, $q \in [2, p]$?

Analyst's Traveling Salesman Problem

P. Jones (1990): Given a set E in a metric space $\mathbb{X}$, decide whether or not E is contained in some rectifiable curve Γ. If so, find a curve $\Gamma \supset E$ "short as possible".

Full solutions for sets in

- $\mathbb{R}^2$ (P. Jones, 1990)
- ℓ^2 (R. Schul, 2007)*
  proof has technical errors, but can be fixed (B-McCurdy, forthcoming)
- Radon measures in $\mathbb{R}^n$ (M. B., R. Schul 2017)
- $\mathbb{R}^n$ (K. Okikiolu, 1992)
- Carnot Groups (S. Li 2019)
- Graph Inverse Limit Spaces (G.C. David, R. Schul 2017)

Partial Survey (Continued)

I. Hahlomaa (2005)

- Sufficient Conditions for $\exists \Gamma \supset E$ in arbitrary metric space $\mathbb{X}$
- Condition is not necessary in $\ell_1^2 = (\mathbb{R}^2, |\cdot|_1)$

G.C. David, R. Schul (2019)

- Necessary Conditions for Γ to be rectifiable in arbitrary metric space $\mathbb{X}$ when Γ doubling

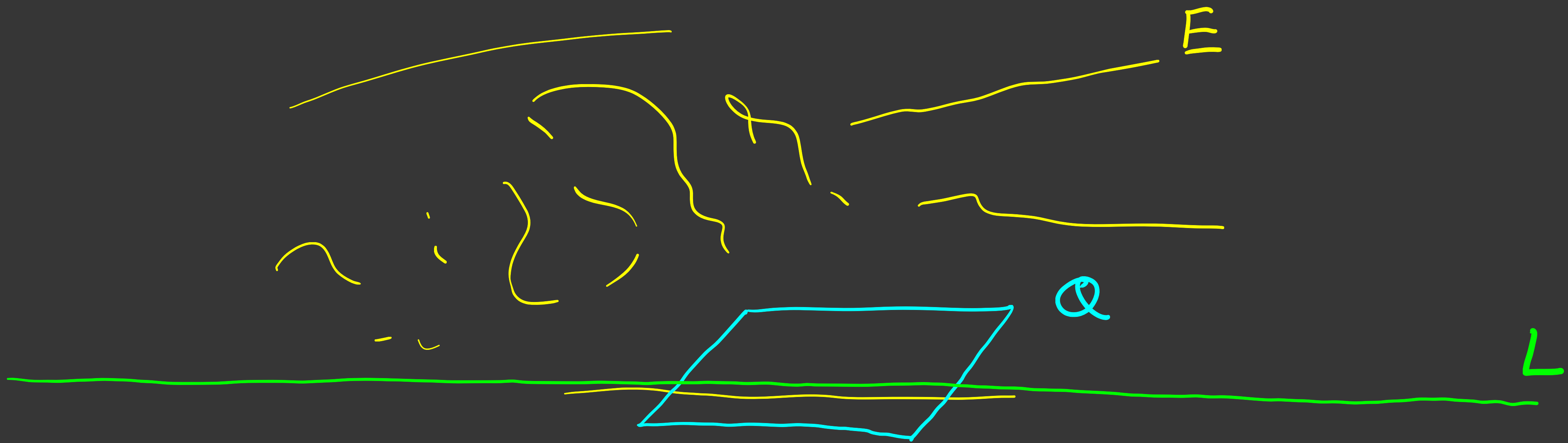
Reifenberg's Algorithm

N. Edelen, A. Naber, D. Valtorta (2019)

- Sufficient Conditions for $\exists$ bi-Lipschitz surface $\supset E$ in ℓ_2 and for $\exists$ bi-Lip curve $\supset E$ in ℓ_p, $1 < p < \infty$

P. Jones β number in a Banach space

"unilateral linear approximation"



set E

window Q

line L

$$\beta_E(Q, L) = \sup_{x \in E \cap Q} \frac{\operatorname{dist}(x, L)}{\operatorname{diam} Q} \in [0,1]$$

$$\beta_E(Q) = \inf_{L} \beta_E(Q, L)$$

Jones-Okikiolu Theorem in Banach spaces

$(\mathbb{X}, |\cdot|)$ finite-dimensional Banach space

Δ system of dyadic cubes (choice of basis)

$E \subset \mathbb{X}$ bounded set

$$\exists\, \Gamma \supset E \text{ with } \mathcal{H}^1(\Gamma) < \infty \quad \text{iff}$$

$$S_E = \sum_{Q \in \Delta} \beta_E(3Q)^2 \cdot \operatorname{diam} Q < \infty$$

Moreover, can find Γ with $\mathcal{H}^1(\Gamma) \approx \operatorname{diam} E + S_E$

where implicit constants only depend on $\dim \mathbb{X}$, Δ (choice of basis) and norm $|\cdot|$

Challenges in Infinite-Dimensions

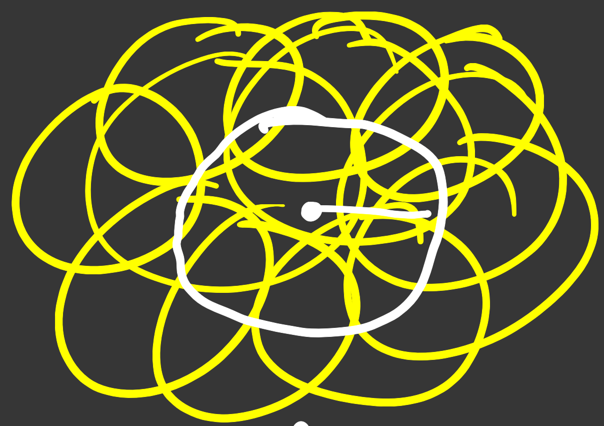
↳ Many Good Ideas by R. Schul

① No "Dyadic Cubes"

↳ Solution: Use 2^{-k}-nets X_k for E and multiresolution families $\{B(x, 3\cdot 2^{-k})\}_{x\in X_k}$

② Uncontrolled Overlap

↳ If $E=\Gamma$ and $\mathcal{H}^1(\Gamma)<\infty$, X_k locally finite but



can be arbitrarily large number of balls $B(y, 3\cdot 2^{-k})$ that intersect $B(x, 3\cdot 2^{-k})$

↳ Soln: Complicated, but use fact when this happens β large

* Proof corrected by B-McCurdy (forthcoming)

Theorem (R. Schul 2007)

$E \subset \ell_2$ bounded is contained in rectifiable curve iff $\sum_{Q \in \mathcal{G}} \beta_E(Q)^2 \operatorname{diam} E < \infty$

↑ Multiresolution Family for E

Theorem (B-McCurdy 2020/2021) $1<p<\infty$

$E \subset \ell_p$ bounded

- If $\sum_{Q \in \mathcal{G}} \beta_E(Q)^{\min(p,2)} \cdot \operatorname{diam} Q < \infty$, then $E \subset \underset{\mathcal{H}^1(\Gamma)<\infty}{\Gamma}$
- If $E \subset \underset{\mathcal{H}^1(\Gamma)<\infty}{\Gamma}$, then $\sum_{Q \in \mathcal{G}} \beta_E(Q)^{\max(p,2)} \operatorname{diam} Q < \infty$

Examples show gap btw $\min(p,2)$ and $\max(p,2)$ cannot be filled in.

↑ Modulus of Smoothness ↑ Modulus of Convexity

Takeaways

① Analyst's TSP

Trying to understand what rectifiable curves and their subsets look like

Quantitative GMT

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② Still Open!

We only have solutions in a few metric spaces

Euclidean/Carnot methods not strong enough

③ Length gain is sensative to direction

In spaces like ℓ_p, $p \neq 2$, we don't understand how to effectively estimate length gain

Beta numbers are not strong enough

Metric Geometry